Journal of Statistical Physics, Vol. 67, Nos. 1/2, 1992

Projection Operator Approach to the Thermodynamic Formalism of Dynamical Systems

Wolfram Just[1]

Received June 10, 1991; final September 3, 1991

An analytical perturbative treatment of characteristic exponents describing the fluctuations of temporal coarse-grained quantities in the context of nonlinear dynamical systems is proposed. It is based on the analysis of the resolvent of the corresponding transfer operator by a projection operator method similar to those used in statistical mechanics. Two different approximation schemes are presented and tested for the case of an exactly solvable but nontrivial model system.

KEY WORDS: Thermodynamic formalism; transfer operator; projection operator technique.

1. INTRODUCTION

The thermodynamic formalism constitutes a program for analyzing the complicated behavior of nonlinear dynamical systems.[1,2] The main quantity that contains a great deal of information about the dynamics is given by the characteristic exponent[3]

$$\Phi^u(q) := \lim_{n\to\infty} \frac{1}{n} \ln \left\langle \exp\left[q \sum_{i=0}^{n-1} u(T^i(x)) \right] \right\rangle \tag{1}$$

where to be definite we restrict the discussion to discrete dynamical systems $x_{n+1} = T(x_n)$. The expectation value in Eq. (1), $\langle\cdots\rangle$, is meant with respect to some distribution of initial points x which is usually assumed to be the (physical) invariant distribution (SRB measure in mathematical terms). The characteristic quantity depends on the function $u(x)$. The case that $u(x)$ is given by the local expansion rate is, by Bowen's theorem,[2] of

[1] Department of Physics, Kyushu University 33, Fukuoka 812, Japan.

0022-4715/92/0400-0271$06.50/0

special importance and is frequently discussed in the literature.[4] In that case the quantity (1) is called the topological pressure. One possibility for the computation of the characteristic exponent is based on the introduction of the transfer operator[5]

$$(\mathscr{H}_q^u h)(x) := \int \delta(x - T(y)) \exp[qu(y)]\, h(y)\, dy \tag{2}$$

Its largest eigenvalue λ_q is connected to the characteristic exponent via the relation $\ln \lambda_q = \Phi^u(q)$. It is the aim of this paper to apply the ideas of the projection operator technique of statistical mechanics[6,7] for an analytical perturbative calculation of the characteristic exponent.

In order to define a projection operator on the space of integrable functions, its dual space has to be introduced in a natural way. Referring back to the representation of the SRB measure proposed by Bowen,[2] it is obvious to consider the space of measures as the dual space and to introduce the bilinear expression

$$(\mu \mid h) := \int h\, d\mu \tag{3}$$

Then the SRB measure μ_{SRB} admits the representation $d\mu_{\text{SRB}} = h_* \, d\nu_*$, where h_* and ν_* denote the eigenelements of the Ruelle–Frobenius–Perron operator $\mathscr{L}$ and its formal adjoint[2] $\mathscr{L}^\dagger$ corresponding to the largest real eigenvalue. In systems admitting an attractor this eigenvalue equals 1, and for the special case of one-dimensional maps, ν_* coincides with the Lebesgue measure, so that the SRB measure is determined by the invariant distribution h_*. By this theorem the dual space structure and the definition (3) are closely related to the dynamical systems under consideration.

In general higher-dimensional systems no relation between $\mathscr{H}_q^u$ and $\mathscr{L}$ is known, to my knowledge. Although such a relation is not necessary for the formal investigations made in this article, it is important for the construction of an appropriate projection operator. For this reason I will concentrate on one-dimensional systems[3] and expanding maps,[8] where the relation $\mathscr{H}_{q=0}^u = \mathscr{L}$ is well established. Also, the function space on which the transfer operator is defined may depend on the properties of the dynamical system. In the case of one-dimensional maps this space is given by piecewise continuous functions.[4]

[2] With respect to the bilinear expression (3).

[3] To be precise, an expansive condition such as $\exists N, \forall x, \prod_{i=0}^{N} |T'(T^{(i)}(x))| > 1$ has to be imposed.

[4] Finite jumps may occur on the countable set of points at which T' is discontinuous and its forward iterates.

With these preliminaries, I will show in the next section how a projection operator formalism for the resolvent of the transfer operator (2) according to the lines of the Mori projection operator technique[7] can be constructed. Section 3 is devoted to the outline of an approximation scheme which is closely connected to the continued fraction expansion for correlation functions in equilibrium statistical mechanics[9] and yields a result that was obtained earlier in a phenomenological way.[10] Section 4 contains an alternative approximation scheme which relies on a perturbation expansion of the memory kernel. Section 5 is devoted to the discussion of simple exactly solvable model system from the viewpoint of these perturbation expansions. Finally, the results are summarized.

2. PROJECTION OPERATOR TECHNIQUE FOR THE RESOLVENT

For the determination of the eigenvalues of the transfer operator, the discussion of the resolvent $(z-\mathscr{H}^u_q)^{-1}$ is a convenient tool. Its singularities define the spectrum of the transfer operator. With the help of the projection operator formalism, this resolvent can be divided into several parts, which allows for a perturbative treatment. Consider a suitably chosen set of relevant functions $\{g_\alpha\}$ and measures $\{\nu_\beta\}$, $\alpha, \beta = 1,..., N$, and define a "projection operator"

$$\mathscr{P} := \sum_{\alpha,\beta} |g_\alpha)(\chi^{-1})_{\alpha\beta}(\nu_\beta|, \qquad \chi_{\beta\alpha} := (\nu_\beta|\, g_\alpha) \tag{4}$$

on the space of integrable functions which admits the usual relations

$$\mathscr{P}^2 = \mathscr{P}, \qquad \mathscr{P} g_\alpha = g_\alpha, \qquad \mathscr{P}^\dagger \nu_\beta = \nu_\beta \tag{5}$$

Due to the fact that Eq. (3) does not represent a scalar product on some Hilbert space, the regularity of the matrix χ is not obvious. This property is based on an appropriate choice of the sets $\{g_\alpha\}$ and $\{\nu_\beta\}$.

Applying the well-known operator identity (ref. 12, p. 367)

$$\mathscr{P}[z-\mathscr{P}\mathscr{H}^u_q\mathscr{P}-\mathscr{P}\mathscr{H}^u_q(z-\mathscr{Q}\mathscr{H}^u_q\mathscr{Q})^{-1}\mathscr{Q}\mathscr{H}^u_q\mathscr{P}]\,\mathscr{P}(z-\mathscr{H}^u_q)^{-1}\mathscr{P}=\mathscr{P},$$
$$\mathscr{Q} := 1-\mathscr{P} \tag{6}$$

to the resolvent $(\nu_\beta|\,(z-\mathscr{H}^u_q)^{-1}\,|g_\alpha)$, one gets with the help of Eqs. (4) and (5) the final result

$$(\nu_\beta|\,(z-\mathscr{H}^u_q)^{-1}\,|g_\alpha) = \left(\frac{1}{z\mathbf{1}-\mathbf{\Omega}-\mathbf{\Gamma}(z)}\,\chi\right)_{\beta\alpha} \tag{7}$$

where the frequency matrix $\mathbf{\Omega}$ and the memory kernel $\mathbf{\Gamma}(z)$ are given by

$$\begin{aligned}\Omega_{\beta\alpha} &:= \sum_{\gamma} (v_\beta | \mathscr{H}^u_q | g_\gamma)(\chi^{-1})_{\gamma\alpha} \\ \Gamma_{\beta\alpha}(z) &:= \sum_{\gamma} (v_\beta | \mathscr{H}^u_q \mathscr{Q} \frac{1}{z - \mathscr{Q}\mathscr{H}^u_q \mathscr{Q}} \mathscr{Q}\mathscr{H}^u_q | g_\gamma)(\chi^{-1})_{\gamma\alpha}\end{aligned} \tag{8}$$

As mentioned above, the poles of the resolvent (7) yield the eigenvalues of the transfer operator (2). Inspecting Eq. (7), one recognizes that this problem has been reduced to the computation of the frequency and memory matrix and solving the algebraic equation $\det(z\mathbf{1} - \mathbf{\Omega} - \mathbf{\Gamma}(z)) = 0$. We show in the next sections that expressions (8) are better adapted to the application of analytical perturbation schemes than the original resolvent.

3. CONTINUED FRACTION EXPANSION

As one is mainly interested in the characteristic exponent (1) if the average is taken with respect to the natural invariant measure, it is of interest to choose the sets $\{g_\alpha\}$ and $\{v_\beta\}$ in such a way that they contain the SRB measure and investigate the poles of the resolvent

$$(v_* | \frac{1}{z - \mathscr{H}^u_q} | h_*) \tag{9}$$

For this reason let us take the smallest sets $\{g_1\} = \{h_*\}$ and $\{v_1\} = \{v_*\}$, $\alpha, \beta = 1$, which fit in with these constraints. Although Eq. (7) yields an exact expression for the resolvent under consideration, it is a hard task to compute the memory kernel $\mathbf{\Gamma}(z)$ in general. Let us therefore go back to an idea of Mori[9] and incorporate the higher order functions $(\mathscr{H}^u_q)^n h_*$ and measures $(\mathscr{H}^{u\dagger}_q)^n v_*$ $(n = 1,..., N)$ into the relevant set. By this procedure a continued fraction expansion for the resolvent (9) can be constructed.

To be definite, take as the relevant sets those which are built up by $(\mathscr{H}^u_q)^n h_*$ and $(\mathscr{H}^{u\dagger}_q)^n v_*$ $(n = 0,..., N)$:

$$\begin{aligned}g_1 &:= h_* \\ v_1 &:= v_* \\ g_{i+1} &:= \left(\prod_{k=1}^{i} \mathscr{Q}_k\right) \mathscr{H}^u_q g_i, \qquad \mathscr{Q}_k := 1 - \frac{|g_k)(v_k|}{(v_k | g_k)}, \qquad i = 1,..., N-1 \\ v_{i+1} &:= \left(\prod_{k=1}^{i} \mathscr{Q}^\dagger_k\right) \mathscr{H}^{u\dagger}_q v_i, \qquad \mathscr{Q}^\dagger_k := 1 - \frac{|v_k)(g_k|}{(v_k | g_k)}, \qquad i = 1,..., N-1\end{aligned} \tag{10}$$

To simplify the considerations, we have chosen orthogonalized sets. In writing down Eq. (10) it has been presupposed that the relations, $(v_i | g_i) \neq 0$, $i = 1,..., N$ hold. For a remark on the opposite case we refer to Appendix B. After a short computation (Appendix A) we obtain the following results for the frequency matrix:

$$\mathbf{\Omega} = \begin{pmatrix} \Omega_{11} & 1 & 0 & \cdots & 0 \\ \Omega_{21} & \Omega_{22} & \ddots & & \vdots \\ 0 & \Omega_{32} & \ddots & 1 & 0 \\ \vdots & & \ddots & \Omega_{N-1,N-1} & 1 \\ 0 & \cdots & 0 & \Omega_{N,N-1} & \Omega_{NN} \end{pmatrix}$$

$$\Omega_{ii} := \frac{(v_i | \mathscr{H}^u_q | g_i)}{(v_i | g_i)}, \qquad \Omega_{i+1,i} := \frac{(v_{i+1} | g_{i+1})}{(v_i | g_i)} \tag{11}$$

and the memory kernel

$$\mathbf{\Gamma}(z) = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 0 \\ 0 & \cdots & 0 & \Gamma_{NN}(z) \end{pmatrix}$$

$$\Gamma_{NN}(z) := (v_N | \mathscr{H}^u_q \mathscr{Q} \frac{1}{z - \mathscr{Q} \mathscr{H}^u_q \mathscr{Q}} \mathscr{Q} \mathscr{H}^u_q | g_N) \tag{12}$$

Inserting these expressions into the general equation (7), one gets (ref. 12, p. 230) for the matrix element (9) the continued fraction expression

$$(v_1 | \frac{1}{z - \mathscr{H}^u_q} | g_1) = \cfrac{1}{z - \Omega_{11} - \cfrac{\Omega_{21}}{z - \Omega_{22} - \cfrac{\Omega_{32}}{\ddots \, \cfrac{}{z - \Omega_{N-1,N-1} - \cfrac{\Omega_{N,N-1}}{z - \Omega_{NN} - \Gamma_{NN}(z)}}}}} \tag{13}$$

This expansion is mainly determined by the elements of the frequency matrix (11), which means by the moments of the transfer operator, which can be computed easily when the SRB measure is known. It coincides with the expansion proposed by Fujisaka and Inoue[10] (Appendix B).

4. PERTURBATION EXPANSION OF THE MEMORY KERNEL

Assume that the transfer operator can be devided into a zeroth-order part $\mathscr{H}_q^0$ which allows for an explict treatment and a small perturbation $\mathscr{H}_q^1$. A perturbative treatment of the eigenvalue problem of the transfer operator along the lines of the Schrödinger perturbation theory used in quantum mechanics[5] is of limited value in the vicinity of q values q_0 at which the operator (2) admits degenerate eigenvalues. Due to small denominators, the radius of convergence of such an expansion decreases as $|q-q_0|$. It is true that around this critical point an expansion in both small quantities $q-q_0$ and $\mathscr{H}_q^1$ is possible, but no expression uniformly valid in q can be obtained by this perturbation scheme in any finite order.

In contrast, it is known that a standard perturbation expansion of the quantities (8) yields a uniformly valid expansion of the resolvent (7) if the projection operator is chosen appropriately.[6,12] In order to perform the formal expansion, let us choose the relevant sets in such a way that the projection operator (4) commutes with the unperturbed part of the transfer operator

$$\mathscr{H}_q^u =: \mathscr{H}_q^0 + \mathscr{H}_q^1, \qquad \mathscr{P}\mathscr{H}_q^0 = \mathscr{H}_q^0 \mathscr{P} \tag{14}$$

The last property can be achieved if one chooses some eigenelements of the zeroth order part as the relevant sets. As we are interested in the behavior of the largest eigenvalue, the eigenelements corresponding to the upper part of the spectrum seem to be the appropriate choice. By virtue of Eq. (14), the definitions (8) read

$$\begin{aligned} \Omega_{\beta\alpha} &= \Omega_{\beta\alpha}^{(0)} + \Omega_{\beta\alpha}^{(1)} = \sum_\gamma (\nu_\beta|\, \mathscr{H}_q^0\, |g_\gamma)(\chi^{-1})_{\gamma\alpha} + \sum_\gamma (\nu_\beta|\, \mathscr{H}_q^1\, |g_\gamma)(\chi^{-1})_{\gamma\alpha} \\ \Gamma_{\beta\alpha}(z) &= \sum_\gamma (\nu_\beta|\, \mathscr{H}_q^1 \mathscr{Q} \frac{1}{z-\mathscr{Q}(\mathscr{H}_q^0+\mathscr{H}_q^1)\mathscr{Q}} \mathscr{Q}\mathscr{H}_q^1\, |g_\gamma)(\chi^{-1})_{\gamma\alpha} \end{aligned} \tag{15}$$

In the lowest nonvanishing order the perturbation in the denominator of the memory kernel can be neglected and one obtains with the help of Eq. (14) the second-order result

$$\Gamma_{\beta\alpha}^{(2)}(z) = \sum_\gamma (\nu_\beta|\, \mathscr{H}_q^1 \mathscr{Q} \frac{1}{z-\mathscr{H}_q^0} \mathscr{Q}\mathscr{H}_q^1\, |g_\gamma)(\chi^{-1})_{\gamma\alpha} \tag{16}$$

Inspecting the expression (16), one recognizes that it contains the projected resolvent $\mathscr{Q}(z-\mathscr{H}_q^0)^{-1}\mathscr{Q}$ instead of the full resolvent of zeroth-order $(z-\mathscr{H}_q^0)^{-1}$ which governs the ordinary perturbation expansion mentioned at the beginning. This difference is crucial and the reason that the expres-

[5] For the perturbation series of non-Hermitian operators see, e.g., ref. 11.

sion (16) is valid beyond the Schrödinger perturbation theory provided that the projection operator is chosen appropriately. To make this statement explicit, let me mention that z takes values in the vicinity of the largest eigenvalue. If $\mathscr{H}_q^0$ allows for eigenvalues which become arbitrarily close to the largest eigenvalue as the parameter q varies, the expression $(z-\mathscr{H}_q^0)^{-1}$ becomes singular even on a subspace which does not contain the eigenvector corresponding to the largest eigenvalue. As a consequence, the radius of convergence of the Schrödinger perturbation expansion tends to zero. On the other hand, the expression $\mathscr{Q}(z-\mathscr{H}_q^0)^{-1}\mathscr{Q}$ remains regular if the eigenelements corresponding to the resonant eigenvalues are incorporated in the projection operator (4). Therefore the expansion presented above is uniformly valid in q and the expression (16) is of second order in the perturbation. Hence one can hope that the frequency matrix determines the main behavior according to Eq. (7).

5. DISCUSSION OF AN EXAMPLE

For the discussion of the approximation schemes presented in the two preceding sections a simple exactly solvable model system that contains nontrivial features is investigated in this section from the viewpoint of the projection operator formalism. First I briefly review the properties of the model system. For a detailed discussion see ref. 13.

Consider the following one-dimensional map:

$$T_\varepsilon(x)=\begin{cases}\gamma x, & x\in[-a,a]\\ -\gamma(x-2a), & x\in(a,\infty),\\ -\gamma(x+2a), & x\in(-\infty,a)\end{cases}\qquad \gamma:=2+\varepsilon \tag{17}$$

which undergoes a symmetry-breaking chaos transition in the limit $\varepsilon\downarrow 0$. The characteristic exponent corresponding to the function

$$u(x)=\begin{cases}1, & x>0\\ -1, & x<0\end{cases} \tag{18}$$

shows a phase transition at $q=0$ in the limit $\varepsilon\downarrow 0$. This transition is governed by the degeneracy of the two largest eigenvalues $\exp(\pm q)$ of the transfer operator (2). At the bifurcation point $\varepsilon=0$ the map $T_{\varepsilon=0}$ admits a simple Markov partition consisting of six intervals,

$$\begin{aligned} I_0^+ &:= [0,a], & I_0^- &:= [-a,0]\\ I_1^+ &:= [a,2a], & I_1^- &:= [-2a,-a]\\ I_2^+ &:= [2a,3a], & I_2^- &:= [-3a,-2a]\end{aligned} \tag{19}$$

On this partition the matrix representation of the transfer operator reads

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \exp(-q) & \exp(-q) & 0 & 0 & \exp(q) \\ 0 & \exp(-q) & \exp(-q) & 0 & 0 & \exp(q) \\ \exp(-q) & 0 & 0 & \exp(q) & \exp(q) & 0 \\ \exp(-q) & 0 & 0 & \exp(q) & \exp(q) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

The right and left eigenvectors corresponding to the eigenvalues $\exp(\pm q)$ yield the following eigenfunctions and eigenmeasures of the transfer operator and its adjoint:

$$\begin{aligned} \lambda_q^{(1)} &= \exp(q) \\ h^{(1)}(x) &= \chi_{I_0^+ \cup I_1^+}(x) \\ d\nu^{(1)} &= \nu^{(1)}(x)\,dx, \qquad \nu^{(1)}(x) = \frac{1}{2a}\left[\chi_{I_0^+ \cup I_1^+}(x) + \exp(-2q)\,\chi_{I_2^-}(x)\right] \end{aligned} \tag{21}$$

$$\begin{aligned} \lambda_q^{(2)} &= \exp(-q) \\ h^{(2)}(x) &= \chi_{I_0^- \cup I_1^-}(x) \\ d\nu^{(2)} &= \nu^{(2)}(x)\,dx, \qquad \nu^{(2)}(x) = \frac{1}{2a}\left[\chi_{I_0^- \cup I_1^-}(x) + \exp(2q)\,\chi_{I_2^+}(x)\right] \end{aligned}$$

where $\chi_I(x)$ denotes the characteristic function of the interval I. The characteristic exponent (1) shows a typical scaling behavior in the vicinity of the bifurcation point. Due to this fact, the investigation of the parameter region $0 < \varepsilon \ll 1$ yields a suitable test for the approximation schemes presented above.

Continued Fraction Expansion. For writing down the expansion (13) the matrix elements (11) have to be computed. We will see that a restriction to second-order terms suffices. From Eqs. (11) and (10) one gets the relations

$$\begin{aligned} \Omega_{11} &= H_1 \\ \Omega_{21} &= H_2 - H_1^2 \\ \Omega_{22}\Omega_{21} &= H_3 - 2H_1 H_2 + H_1^3 \\ \Omega_{32}\,\Omega_{21}^2 &= (H_4 - H_2^2)(H_2 - H_1^2) - (H_3 - H_1 H_2)^2 \end{aligned} \tag{22}$$

where

$$H_i := (v_*|\,(\mathscr{H}_q^u)^i\,|h_*) \tag{23}$$

denote the moments of the transfer operator with respect to the SRB measure. For the special model system under consideration these expressions can be computed easily even without having explicit knowledge of the invariant distribution (Appendix C). Finally one gets for the matrix elements (22)

$$\begin{aligned}
\Omega_{11} &= \cosh(q) \\
\Omega_{21} &= (1-4\delta)\sinh^2(q) \\
\Omega_{22} &= \frac{1-8\delta}{1-4\delta}\cosh(q) \\
\Omega_{32} &= -\frac{-16\delta^2}{(1-4\delta)^2} - \frac{32\delta^2(1-2\delta)}{(1-4\delta)^2}\sinh^2(q)
\end{aligned} \tag{24}$$

Here δ denotes the SRB measure of the interval $(2a, \infty)$. It has the meaning of the transition rate from the region $x>0$ to the region $x<0$ and is a small quantity of order ε (see Appendix C). As the matrix element (24_4) is an order of magnitude smaller than the other ones, a truncation of the continued fraction after the second term is justified in the limit of small ε. Then the resolvent reads

$$(v_1|\,\frac{1}{z-\mathscr{H}_q^u}\,|g_1) \simeq \frac{1}{z-\Omega_{11}-\Omega_{21}/(z-\Omega_{22})} \tag{25}$$

which was called a two-pole approximation in ref. 10. The singularities of this expression which determine the eigenvalues of the transfer operator can be calculated easily and one gets the correct scaling behavior for the eigenvalues in the vicinity of the phase transition point[13]

$$q =: \frac{4\delta}{2}\kappa, \qquad \ln\lambda_q^{(1/2)} =: \frac{4\delta}{2}\psi_q^{(1/2)} \qquad \psi_q^{(1/2)} \simeq -1 \pm (1+\kappa^2)^{1/2} \tag{26}$$

Perturbation Expansion. For the perturbative treatment we take the transfer operator corresponding to the map $T_{\varepsilon=0}$ as the unperturbed part. Referring back to Section 4, we build up the relevant sets with the eigenelements (21) which belong to a part of the spectrum of $\mathscr{H}_q^0$ that is well separated from the remaining one.[13] With Eqs. (15) and (21) a simple straightforward calculation yields the frequency matrix

$$\Omega_{11}=\iint v^{(1)}(x)\,\delta(x-T_\varepsilon(y))\,h^{(1)}(y)\,dy\,dx=\frac{2}{\gamma}\exp(q)$$
$$\Omega_{12}=\iint v^{(1)}(x)\,\delta(x-T_\varepsilon(y))\,h^{(2)}(y)\,dy\,dx=\frac{\varepsilon}{\gamma}\exp(-3q)$$
$$\Omega_{21}=\iint v^{(2)}(x)\,\delta(x-T_\varepsilon(y))\,h^{(1)}(y)\,dy\,dx=\frac{\varepsilon}{\gamma}\exp(3q) \tag{27}$$
$$\Omega_{22}=\iint v^{(2)}(x)\,\delta(x-T_\varepsilon(y))\,h^{(2)}(y)\,dy\,dx=\frac{2}{\gamma}\exp(-q)$$

Referring to the discussion in Section 4, it seems reasonable that the memory matrix is regular and of second order in the vicinity of the phase transition point $q=0$, $z=1$. It is indeed possible to derive the following estimate:

$$\boldsymbol{\Gamma}^{(2)}(z)=O(\varepsilon^{2-\eta}), \qquad \text{if} \quad |z|>\exp(|q|-\eta\ln 2), \qquad \eta>0 \text{ arbitrary} \tag{28}$$

I devote Appendix D to the tedious computation. Inspecting Eq. (7), one sees that the poles of the resolvent are determined by the eigenvalues of the frequency matrix

$$\boldsymbol{\Omega}^{(0)}+\boldsymbol{\Omega}^{(1)}=\begin{pmatrix}(1-\frac{1}{2}\varepsilon)\exp(q) & \frac{1}{2}\varepsilon\exp(-3q)\\ \frac{1}{2}\varepsilon\exp(3q) & (1-\frac{1}{2}\varepsilon)\exp(-q)\end{pmatrix} \tag{29}$$

up to the lowest nonvanishing order in the small parameter ε. Introducing again the abbreviations $q=\kappa\varepsilon/2$ and $\ln\lambda_q=\psi_q\varepsilon/2$, one recovers the scaling behavior (26).

6. SUMMARY

Based on the projection operator approach used in statistical mechanics, the resolvent of the transfer operator has been investigated. The computation of the characteristic exponent has been reduced to the determination of the frequency and memory matrix (8). For the construction of a projection operator we have referred back to a dual space structure which is naturally connected to the dynamics generated by T_ε. Two approximation schemes have been proposed, which on one hand allow for an explicit evaluation of these quantities and on the other hand define the relevant functions and measures which determine the projection operator. The first approach starts from the minimal set containing the SRB measure and extends it in a systematic way. By this procedure a continued fraction expansion of the resolvent has been obtained which involves the moments of the transfer operator. The second one is based on a perturbative approach starting from an unperturbed part of the transfer operator. The relevant sets are determined by the upper part of the spectrum of this operator, which should be separated from the remainder. Also in this case only simple matrix elements of the transfer operator have to be evaluated. Both schemes have been applied to a simple model system exhibiting a symmetry-breaking bifurcation. The correct behavior in the vicinity of the phase transition point was reproduced. Additionally both approaches lead to a systematic expansion in the relevant small parameter. From these results we draw the conclusion that the proposed formalism may be helpful for a perturbative treatment of more complicated dynamical systems.

APPENDIX A

From the definition (10) we get the orthogonality relations

$$\chi_{ij} = (v_i \mid g_j) = \delta_{ij}(v_i \mid g_i) \tag{A1}$$

From the same equation we obtain that $\mathscr{H}^u_q g_i$ ($\mathscr{H}^{u\dagger}_q v_i$) can be represented by a sum of the elements $\{g_1,..., g_{i+1}\}$ ($\{v_1,..., v_{i+1}\}$). Referring to the definition (8_1) and to Eq. (A1), we see that the nonvanishing elements of the frequency matrix read

$$\Omega_{i-1,i} = \frac{(v_{i-1} \mid \mathscr{H}^u_q \mid g_i)}{(v_i \mid g_i)}, \qquad \Omega_{ii} = \frac{(v_i \mid \mathscr{H}^u_q \mid g_i)}{(v_i \mid g_i)}, \qquad \Omega_{i+1,i} = \frac{(v_{i+1} \mid \mathscr{H}^u_q \mid g_i)}{(v_i \mid g_i)} \tag{A2}$$

Multiplying Eq. (10_3) with $(v_{i+1}|$ and Eq. (10_4) with $|g_{i+1})$, we obtain with the help of the orthogonality relation (A1)

$$\begin{aligned} (v_{i+1} \mid g_{i+1}) &= (v_{i+1} \mid \mathscr{H}^u_q \mid g_i) \\ (v_{i+1} \mid g_{i+1}) &= (\mathscr{H}^{u\dagger}_q v_i \mid g_{i+1}) \end{aligned} \tag{A3}$$

Combining Eqs. (A2) and (A3) yields Eq. (11).

From the remark made at the beginning of the Appendix, we get immediately

$$\mathscr{Q}\mathscr{H}^u_q g_i = 0, \qquad \mathscr{Q}^\dagger \mathscr{H}^{u\dagger}_q v_i = 0, \qquad i = 1,..., N-1 \tag{A4}$$

From this relation the result (12) is obvious.

APPENDIX B

To show the identity to the continued fraction expansion proposed in Ref. 10, we introduce the resolvent matrix

$$R_{\beta\alpha}(z) := (v_\beta|\ (z - \mathscr{H}^u_q)^{-1}\ |g_\alpha) \tag{B1}$$

Then Eq. (7) yields

$$\delta_{i1}(v_1 \mid g_1) = ([z\mathbf{1} - \boldsymbol{\Omega} - \boldsymbol{\Gamma}(z)]\ \mathbf{R}(z))_{i1} \tag{B2}$$

Using the expressions (11) and (12) and the abbreviation

$$\Psi_i(z) := \frac{R_{i+1,1}(z)}{R_{i1}(z)} \tag{B3}$$

we obtain for Eq. (B2)

$$\begin{aligned}(v_1 \mid g_1) &= [z - \Omega_{11} - \Psi_1(z)]\ R_{11}(z)\\ 0 &= -\Omega_{i,i-1} + [z - \Omega_{ii} - \Psi_i(z)]\ \Psi_{i-1}, \qquad i = 2,..., N-1\\ 0 &= -\Omega_{N,N-1} + [z - \Omega_{NN} - \Gamma_{NN}(z)]\ \Psi_{N-1}\end{aligned} \tag{B4}$$

To get the formal equivalence, one has to introduce the notation

$$\mu := \frac{1}{z}, \qquad [\mu]_0 := zR_{11}(z), \qquad [\mu]_i := z\Psi_i(z), \qquad i = 1,..., N-1 \tag{B5}$$

Then Eqs. $(\mathrm{B4}_{1,2})$ become

$$\begin{aligned}(v_1 \mid g_1) &= (1 - \mu\Omega_{11} + \mu^2[\mu]_1)[\mu]_0\\ \Omega_{i,i-1} &= (1 - \mu\Omega_{ii} - \mu^2[\mu]_i)[\mu]_{i-1}\end{aligned} \tag{B6}$$

which is identical to Eq. (10) of ref. 10.

The formal continued fraction expansion stops if at some stage the expression $(v_i \mid g_i)$ vanishes. Let us therefore assume that the construction of Section 3 holds up to the order N and denote the relevant sets by $\{g_1,..., g_N\}$ and $\{v_1,..., v_N\}$. Furthermore, the construction should fail at the next level, which means

$$g_c := \prod_{k=1}^{N} \mathscr{Q}_k \mathscr{H}^u_q g_N, \qquad v_c := \prod_{k=1}^{N} \mathscr{Q}_k^\dagger \mathscr{H}^{u\dagger}_q v_N, \qquad (v_c \mid g_c) = 0 \tag{B7}$$

There is no unique way to circumvent this difficulty. One possible way is to omit the expressions (B7) from the relevant set and incorporate in the next step the following function and measure:

$$g_{N+1} := \prod_{k=1}^{N} \mathscr{Q}_k \mathscr{H}^u_q g_c, \qquad v_{N+1} := \prod_{k=1}^{N} \mathscr{Q}_k^\dagger \mathscr{H}^{u\dagger}_q v_c \tag{B8}$$

provided $(v_{N+1} | g_{N+1}) \neq 0$. Also, with these sets it is possible to write down a continued fraction expansion and extend it to higher orders. But as the relation (A4) ceases to be valid for $i = N$, this expansion does not share the simple structure of Eq. (13). We therefore omit the explicit expressions.

APPENDIX C

Consider the sets

$$\begin{aligned} J_{+1} &:= \{x \mid x > 0\}, \qquad J_{-1} := \{x \mid x < 0\} \\ J_{i_0,\dots,i_{k-1}} &:= \{x \mid T_\varepsilon^j(x) \in J_{i_j},\, 0 \leqslant j \leqslant k-1\} \end{aligned} \tag{C1}$$

Then the moments (23) of the transfer operator read

$$\begin{aligned} H_k &= \int \exp\left[q \sum_{j=0}^{k-1} u(T_\varepsilon^j(x))\right] h_*(x)\, dx \\ &= \sum_{i_0,\dots,i_{k-1}} \exp\left(q \sum_{j=0}^{k-1} i_j\right) \int_{J_{i_0,\dots,i_{k-1}}} h_*(x)\, dx \\ &= \sum_{i_0,\dots,i_{k-1}} \exp\left(q \sum_{j=0}^{k-1} i_j\right) \mu_{i_0,\dots,i_{k-1}} \end{aligned} \tag{C2}$$

Here the fact that ν_* coincides with the Lebesgue measure and the abbreviation $\mu_{i_0,\dots,i_{k-1}}$ for the SRB measure of the sets $(\mathrm{C}1_2)$ have been used. To compute the last mentioned quantity, we note that

$$\begin{aligned} &\bigcup_{i_k} J_{i_0,\dots,i_k} = J_{i_0,\dots,i_{k-1}} \\ &\bigcup_{i_0} J_{i_0,\dots,i_k} = T_\varepsilon^{-1}(J_{i_1,\dots,i_k}), \qquad k \geqslant 1 \end{aligned} \tag{C3}$$

so that the T_ε invariance of the SRB measure leads to

$$\begin{aligned} &\sum_{i_k} \mu_{i_0,\dots,i_k} = \mu_{i_0,\dots,i_{k-1}} \\ &\sum_{i_0} \mu_{i_0,\dots,i_k} = \mu_{i_1,\dots,i_k}, \qquad k \geqslant 1 \end{aligned} \tag{C4}$$

By the symmetry of the map, we have

$$\mu_+ = \mu_{-1} = \tfrac{1}{2} \tag{C5}$$

If ε is sufficiently small, every trajectory leaving the interval J_{+1} (J_{-1}) stays in the other interval J_{-1} (J_{+1}) for a sufficiently long time. This means

$$
\begin{aligned}
0 &= \mu_{+1,-1,+1} = \mu_{-1,+1,-1} \\
0 &= \mu_{+1,-1,+1,+1} = \mu_{-1,+1,-1,-1} \\
0 &= \mu_{+1,+1,-1,+1} = \mu_{-1,-1,+1,-1} \\
0 &= \mu_{-1,+1,+1,-1} = \mu_{+1,-1,-1,+1}
\end{aligned}
\tag{C6}
$$

The remaining quantities can be determined uniquely from Eqs. (C4) in the case $k = 1, 2, 3$:

$$
\begin{aligned}
\delta &:= \mu_{+1,-1} = \mu_{-1,+1} \\
\tfrac{1}{2} - \delta &= \mu_{+1,+1} = \mu_{-1,-1} \\
\delta &= \mu_{+1,-1,-1} = \mu_{-1,+1,+1} = \mu_{+1,+1,-1} = \mu_{-1,-1,+1} \\
\tfrac{1}{2} - 2\delta &= \mu_{+1,+1,+1} = \mu_{-1,-1,-1} \\
\delta &= \mu_{+1,+1,-1,-1} = \mu_{-1,-1,+1,+1} \\
\delta &= \mu_{+1,+1,+1,-1} = \mu_{-1,-1,-1,+1} \\
\delta &= \mu_{+1,-1,-1,-1} = \mu_{-1,+1,+1,+1} \\
\tfrac{1}{2} - 3\delta &= \mu_{+1,+1,+1,+1} = \mu_{-1,-1,-1,-1}
\end{aligned}
\tag{C7}
$$

Here the quantity defined in Eq. ($C7_1$) measures those points which leave one of the intervals ($C1_1$) and can be identified with the escape rate. Referring to the tentlike shape of the mapping (17), one can estimated it as $\delta \simeq \varepsilon a/(4a) = \varepsilon/4$.[6] Inserting Eqs. (C5)–(C7) into Eq. (C2), we obtain

$$
\begin{aligned}
H_1 &= \cosh(q) \\
H_2 &= \cosh(2q) + 2\delta[1 - \cosh(2q)] \\
H_3 &= \cosh(3q) + 4\delta[\cosh(q) - \cosh(3q)] \\
H_4 &= \cosh(4q) + 2\delta[-3\cosh(4q) + 2\cosh(2q) + 1]
\end{aligned}
\tag{C8}
$$

With Eq. (22), one gets the result (24).

APPENDIX D

The matrix element $\Gamma_{11}(z)$ will be evaluated explicitly. The other elements can be calculated in the same way.

From the definition (2) and (21_2) we obtain

$$
(\mathscr{H}^u_q h^{(1)})(x) = \frac{2}{\gamma} e^q [\chi_{I_0^+ \cup I_1^+}(x) + \chi_{[2a,\gamma a]}(x)] \tag{D1}
$$

[6] The exact expression reads $\delta = \varepsilon/(2\gamma)$.

so that by the definition (4) of the projection operator the relation

$$(\mathscr{Q}\mathscr{H}_q^1 h^{(1)})(x) = (\mathscr{Q}\mathscr{H}_q^u h^{(1)})(x) = \frac{2}{\gamma} e^q \mathscr{Q}\chi_{[2a,\gamma a]}(x) \tag{D2}$$

holds.

If v is a measure with density $v(x)$, then by the definition of the adjoint operator $\mathscr{H}_q^{u\dagger}$ the measure $\mathscr{H}_q^{u\dagger}v$ is determined by the density $\exp[qu(x)]\ v(T_\varepsilon(x))$. This result follows by a straightforward computation. Using this relation, one gets

$$(\mathscr{H}_q^{u\dagger}v^{(1)})(x) = \frac{e^q}{2a}[v^{(1)}(x) - \chi_{K^+}(x)] + \frac{e^{-3q}}{2a}\chi_{K^-}(x) \tag{D3}$$

where

$$K^+ := \left[\frac{\gamma}{2a}, 2a - \frac{\gamma}{2a}\right], \qquad K^- := \left[-2a + \frac{\gamma}{2a}, -\frac{\gamma}{2a}\right] \tag{D4}$$

denote those parts of the intervals $I_0^+ \cup I_1^+$ and $I_0^- \cup I_1^-$ that are mapped by T_ε on the intervals I_2^+ and I_2^-. From Eq. (D3) we obtain immediately

$$(\mathscr{Q}^\dagger \mathscr{H}_q^{1\dagger} v^{(1)})(x) = \frac{1}{2a}\mathscr{Q}^\dagger[e^{-3q}\chi_{K^-}(x) - e^q\chi_{K^+}(x)] \tag{D5}$$

With Eqs. (D2) and (D5), Eq. (16) yields for the element under consideration

$$\Gamma_{11}(z) = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}\int \frac{1}{2a}[e^{-3q}\chi_{K^-}(x) - e^q\chi_{K^+}(x)]\ \mathscr{Q}(\mathscr{H}_q^0)^k \frac{2}{\gamma} e^q \chi_{[2a,\gamma a]}(x)\, dx \tag{D6}$$

For the following considerations we will assume that for a sufficiently large natural number N the relation $\varepsilon = 2^{-N+1}$ holds. This assumption restricts the ε values to a countable sequence. But as this sequence tends to 0, it is reasonable to assume that it contains the main behavior for small ε values. We then obtain

$$\begin{aligned} ((\mathscr{H}_q^0)^k \chi_{[2a,\gamma a]})(x) &= \frac{e^q}{2}\left(\frac{e^{-q}}{2}\right)^{k-1} \chi_{[-2^k\varepsilon a, 0]}(x), \qquad 1 \leqslant k \leqslant N \\ ((\mathscr{H}_q^0)^k \chi_{[2a,\gamma a]})(x) &\sim \chi_{I_0^- \cup I_1^-}(x), \qquad k > N \end{aligned} \tag{D7}$$

where the first relation follows by induction and the second from the requirement given above. With these relations the integrand in Eq. (D6) can be evaluated. One recognizes that in the case $k>N$ the summands are eliminated by the projection operator. For the remaining part we get, by using the definition (4),

$$\begin{aligned}\Gamma_{11}(z)=&\frac{e^q}{z\gamma a}\int [e^{-3q}\chi_{K^-}(x)-e^q\chi_{K^+}(x)]\,\mathcal{Q}\chi_{[2a,\gamma a]}(x)\,dx\\&+\frac{e^q}{z\gamma a}\sum_{k=1}^{N-1} e^{2q}\left(\frac{e^{-q}}{2z}\right)^k\int [e^{-3q}\chi_{K^-}(x)-e^q\chi_{K^+}(x)]\,\mathcal{Q}\chi_{[-2^k\varepsilon a,0]}(x)\,dx\\=&\frac{e^q}{z\gamma a}\left(-\frac{\varepsilon^2 a e^{-q}}{\gamma}\right)+\frac{e^q}{z\gamma a}\sum_{k=1}^{N-2} e^{2q}\left(\frac{e^{-q}}{2z}\right)^k\left(-\frac{\varepsilon^2 a 2^k e^{-3q}}{\gamma}\right)\end{aligned}\tag{D8}$$

Finally we obtain

$$\Gamma_{11}(z)=-\frac{\varepsilon^2}{\gamma^2}\frac{1}{z}\sum_{k=0}^{N-2}\left(\frac{e^{-q}}{z}\right)^k,\qquad \varepsilon=2^{-N+1}\tag{D9}$$

The other matrix elements can be calculated in a similar fashion, leading to the result

$$\begin{aligned}\Gamma_{12}(z)&=\frac{\varepsilon^2}{\gamma^2}\frac{e^{-2q}}{z}\sum_{k=0}^{N-2}\left(\frac{e^{q}}{z}\right)^k\\\Gamma_{21}(z)&=\frac{\varepsilon^2}{\gamma^2}\frac{e^{2q}}{z}\sum_{k=0}^{N-2}\left(\frac{e^{-q}}{z}\right)^k\\\Gamma_{22}(z)&=-\frac{\varepsilon^2}{\gamma^2}\frac{1}{z}\sum_{k=0}^{N-2}\left(\frac{e^{q}}{z}\right)^k\end{aligned}\tag{D10}$$

Using the estimate

$$\begin{aligned}\left|\varepsilon^2\sum_{k=0}^{N-2}\left(\frac{e^{\pm q}}{z}\right)^k\right|&=\varepsilon^{2-\eta}\left|\sum_{k=0}^{N-2}\left(\frac{e^{\pm q}}{z}\right)^k 2^{-\eta(N-1)}\right|\\&\leqslant\varepsilon^{2-\eta}\sum_{k=0}^{\infty}\left|\frac{e^{\pm q}}{2^\eta z}\right|^k\end{aligned}\tag{D11}$$

one recognizes that the matrix elements (D9) and (D10) are of $O(\varepsilon^{2-\eta})$ if the series on the right-hand side of Eq. (D11) converges.

ACKNOWLEDGMENTS

The author is greatly indebted to the Japan Society for Promotion of Science for financial support through a postdoctoral fellowship and to the

Alexander von Humboldt Stiftung for financial support through the Feodor Lynen program. Part of this work was performed within a program of the Sonderforschungsbereich 185 Darmstadt-Frankfurt at the TH Darmstadt, Germany.

REFERENCES

1. D. Ruelle, *Encyclopedia of Mathematics and Its Applications*, Vol. 5, *Thermodynamic Formalism* (Addison-Wesley, Reading, Massachusetts, 1978); T. Morita, H. Hata, H. Mori, T. Horita, and K. Tomita, *Prog. Theor. Phys.* **79**:296 (1988); P. Grassberger, R. Badii, and A. Politi, *J. Stat. Phys.* **51**:135 (1988).
2. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, Berlin, 1975).
3. H. Fujisaka, *Prog. Theor. Phys.* **70**:1264 (1983); H. Fujisaka and M. Inoue, *Prog. Theor. Phys.* **77**:1334 (1987).
4. K. Tomita, H. Hata, T. Horita, H. Mori, and T. Morita, *Prog. Theor. Phys.* **80**:953 (1988); D. Bessis, G. Paladin, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **51**:109 (1988); M. J. Feigenbaum, I. Procaccia, and T. Tel, *Phys. Rev. A* **39**:5359 (1989).
5. P. Szepfalusy and T. Tel, *Phys. Rev. A* **34**:2520 (1986); H. Fujisaka and M. Inoue, *Prog. Theor. Phys.* **78**:268 (1987).
6. R. Zwanzig, *J. Chem. Phys.* **33**:1338 (1960).
7. H. Mori, *Prog. Theor. Phys.* **33**:423 (1965); H. Fujisaka and T. Yamada, *Z. Naturforsch.* **33A**:1455 (1978).
8. P. Walters, *Trans. Am. Math. Soc.* **236**:121 (1978).
9. H. Mori, *Prog. Theor. Phys.* **34**:399 (1965).
10. H. Fujisaka and M. Inoue, *Prog. Theor. Phys.* **78**:1203 (1987).
11. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1976).
12. E. Fick and G. Sauermann, *The Quantum Statistics of Dynamic Processes* (Springer, Berlin, 1990).
13. W. Just, *Phys. Lett. A* **150**:362 (1990).